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Factorization in integrable systems with impurity

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Abstract

This article is based on recent works done in collaboration with M. Mintchev, É. Ragoucy and P. Sorba. It aims at presenting the latest developments in the subject of factorization for integrable field theories with a reflecting and transmitting impurity.

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Introduction

One particular aspect of quantum integrable field theories is the existence, by definition, of countably many independent conserved quantities. In $1+1$ dimensions, nontrivial massive quantum field theories can be studied non-perturbatively thanks to the following results [1]: there is no particle production, the momenta are conserved individually and finally, any N-particle process can be decomposed as a sequence of two-particle processes. This last property is the so-called *factorization property*. It follows that the central ingredient for such theories is the two-body scattering matrix which has to satisfy the celebrated (quantum) Yang-Baxter equation [2, 3].

It is the purpose of this article to present in a comprehensive way how these well-known facts can be generalized to account for the presence of a reflecting and transmitting defect, called impurity. To this end, we will briefly recast what is known of factorization for integrable field theories on the whole line and on the half-line. When including an impurity, this has been generalized in two different ways, resulting in an apparent difficulty. We will show how to reconcile the two points of view.

1 Factorization on the whole line

1.1 Physical data and Yang-Baxter equation

Let us consider a quantum integrable field theory for massive particles with n internal degrees of freedom. One of the essential ingredients is the two-body scattering matrix whose coefficients $\mathcal{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(k_1,k_2)$, $\alpha_1,\ldots,\beta_2=1,\ldots,n$ are functions of the rapidities (or momenta) $k_1,\ k_2\in\mathbb{C}$ parametrizing the dispersion relation of the two particles. This matrix encodes the interaction between particles. For convenience, in the rest of the article, we will use auxiliary spaces and consider the two-body scattering matrix $\mathcal{S}_{12}(k_1,k_2)$ as an element of $End(\mathbb{C}^n\otimes\mathbb{C}^n)(k_1,k_2)$

$$\mathcal{S}_{12}(k_1,k_2)=\mathcal{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(k_1,k_2)\ E_{\alpha_1\beta_1}\otimes E_{\alpha_2\beta_2}\,, \tag{1.1}$$

where $E_{\alpha\beta}$, $\alpha,\beta=1,\ldots,n$ is the canonical basis of $\mathbb{C}^n$ and summation over repeated indices is implied.

The physical unitarity of the total scattering matrix is guaranteed by that of the two-body scattering matrix

$$\mathcal{S}_{12}(k_1,k_2)\mathcal{S}_{12}^\dagger(k_1,k_2)=\mathbb{I}\otimes\mathbb{I}\,, \tag{1.2}$$

where the dagger stands for Hermitian conjugation and $\mathbb{I}$ is the $n\times n$ unit matrix. In turn, this is implied by two conditions that are usually imposed for convenience

$$\text{Unitarity}\qquad \mathcal{S}_{12}(k_1,k_2)\mathcal{S}_{21}(k_2,k_1)=\mathbb{I}\otimes\mathbb{I}\,, \tag{1.3}$$

$$\text{Hermitian analyticity}\qquad \mathcal{S}_{12}^\dagger(k_1,k_2)=\mathcal{S}_{21}(k_2,k_1)\,. \tag{1.4}$$

In this context, the factorization property of the underlying theory is represented by the Yang-Baxter equation that we require for the two-body scattering matrix

$$\mathcal{S}_{12}(k_1,k_2)\mathcal{S}_{13}(k_1,k_3)\mathcal{S}_{23}(k_2,k_3) = \mathcal{S}_{23}(k_2,k_3)\mathcal{S}_{13}(k_1,k_3)\mathcal{S}_{12}(k_1,k_2)\,. \tag{1.5}$$

This has to be understood as an identity in $End(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)(k_1, k_2, k_3)$.

1.2 Algebraic setup

It has been realized in [4, 5] that the previous two-body scattering matrix could be taken as the central piece of an algebraic setup analogous to the Heisenberg algebra but aiming at describing the asymptotic states of an *interacting* theory characterized by $\mathcal{S}_{12}(k_1,k_2)$. The consistency of this approach is ensured by the above conditions imposed on $\mathcal{S}_{12}(k_1,k_2)$. This is known as the Zamolodchikov-Faddeev (ZF) algebra [4, 5]. This approach has been rigorously investigated in [6] where the Fock space representation of the ZF algebra was explicitly constructed. The ZF algebra has proved extremely fruitful in the Quantum Inverse Scattering method (see e.g. [5]).

In this context, the quantum nonlinear Schrödinger equation is the paradigm of quantum integrable field theories (see e.g. [7] and references therein). Indeed, using the ZF algebra for this system, it is possible to reconstruct the canonical quantum field satisfying the nonlinear evolution equation. One also easily gets the N-particle scattering matrix elements and the time-dependent correlation functions (at zero temperature). Finally, this algebraic approach enables one to identify the symmetry algebra of the system [8, 9]. This powerful approach makes it very tempting to generalize the algebraic setup when investigating systems with boundary or impurity.

2 Generalizing factorization

Consider a quantum integrable system with a (fixed) boundary. We are interested in the factorization property of the theory knowing that, in addition to the two-body scattering matrix $\mathcal{S}_{12}(k_1,k_2)$, one has to take into account the reflection matrix $\mathcal{R}(k)$. The latter represents the scattering properties of a particle characterized by k with the boundary. Generalizing the ideas of [4] to a system with boundary, I. Cherednik discovered in [10] that the reflection matrix should satisfy the *reflection equation*. The factorization property means here that the N-particle scattering matrix is built out of $\mathcal{S}_{12}(k_1,k_2)$ and $\mathcal{R}(k)$ only. In analogy with the properties listed in the first section for $\mathcal{S}_{12}(k_1,k_2)$ one requires the following properties for the physical data $\mathcal{R}(k)$

Unitarity
$$\mathcal{R}(k)\mathcal{R}(-k) = \mathbb{I}\,, \tag{2.1}$$

Hermitian analyticity
$$\mathcal{R}^\dagger(k) = \mathcal{R}(-k)\,, \tag{2.2}$$

Reflection equation
$$\begin{aligned}&\mathcal{S}_{12}(k_1,k_2)\mathcal{R}_1(k_1)\mathcal{S}_{21}(k_2,-k_1)\mathcal{R}_2(k_2)\\ &= \mathcal{R}_2(k_2)\mathcal{S}_{12}(k_1,-k_2)\mathcal{R}_1(k_1)\mathcal{S}_{21}(-k_2,-k_1)\,.\end{aligned} \tag{2.3}$$

2.1 Algebraic point of view

As explained above, it is very promising to design an algebraic approach for systems with boundary. Two approaches have been developed so far. The first one is based on a *boundary operator* [11, 12] which is added to the usual ZF generators. The second and more recent approach relies on the so-called *boundary algebra* [13] which also includes an additional generator accounting for the boundary but deeply modifies the defining relations of the original ZF algebra. Each approach has its own interests. For example, the first one has been convenient for the bootstrap program with boundary (see e.g. [11, 14]) while the second is most useful in the quantum inverse scattering method with boundary (see e.g. [15]). Let us stress however that there is no rigorous justification of the first approach (as admitted by the authors of [11] themselves) as opposed to the second one [13].

Nevertheless, there is no difference between the two approaches at the level of the equations imposed on the physical data. Both consistently reproduce the sets of relations (1.3-1.5) and (2.1-2.3).

Both approaches were naturally generalized to the case of a defect (or impurity). In this case, the possibility for transmission is encoded in an additional matrix: the transmission matrix $\mathcal{T}(k)$. At the algebraic level, the differences are even more important than in the boundary case. Indeed, in the first approach, the boundary operator is replaced by a *single defect operator* added to the usual ZF algebra, giving rise to a *defect algebra* [16] while in the second approach, the boundary algebra is turned into a *reflection-transmission algebra* where *two* additional generators account for the presence of the impurity [17]. The status of the two approaches is parallel to the boundary case. The first one remains quite formal and has been used for various computations of statistical physics [16]. Let us stress however that a strong restriction discovered in [18] holds in this context, limiting the study of integrable systems with reflection and transmission essentially to noninteracting ones. The second approach stands on mathematical foundations involving the explicit construction of Fock representations [17]. It proved fundamental in the quantum inverse scattering method applied to the nonlinear Schrödinger equation with impurity [19], which constitutes the first known example of this kind, and in the investigation of the corresponding symmetry algebra [20].

Nevertheless, the situation is completely different from the boundary case as soon as one is interested in the factorization property for the physical data since the two approaches seem to give different physical equations. This is explained in the rest of this article.

2.2 Going back to physical data

We are concerned with the analogs of the Yang-Baxter and reflection equations when transmission is allowed. Let us emphasize that these *reflection-transmission quantum Yang-Baxter equations* (RTQYBE) are the crucial elements for physics

in the sense that they encode the factorization property of any N-particle process and allow for physical computations. The previous algebraic setups are convenient theoretical tools for which the only constraints are self-consistency and consistent reproduction of the RTQYBE.

Let us focus now on the equations obtained in the first approach. To be accurate, introduce the Lorentz (Galilean) invariant two-body scattering matrix $S_{12}(k_1-k_2)$. Introduce also two reflection matrices $R^+(k)$, $R^-(k)$ and two transmission matrices $T^+(k)$, $T^-(k)$, as is done in [18]. They account for possible different bahaviours on the left and on the right of the impurity. Invoking the consistency and the associativity of the corresponding defect algebra, one recovers the fundamental relations (1.3-1.5) and deduces the following typical relations

$$\begin{aligned} S_{12}(k_1-k_2)R_1^+(-k_1)S_{21}(k_1+k_2)R_2^+(-k_2) \\ = R_2^+(-k_2)S_{12}(k_1+k_2)R_1^+(-k_1)S_{21}(k_1-k_2)\,, \end{aligned} \tag{2.4}$$

$$T_1^+(k_1)S_{21}(k_2-k_1)R_2^+(-k_2) = R_2^+(-k_2)T_1^+(k_1)S_{21}(-k_2-k_1)\,, \tag{2.5}$$

$$S_{12}(k_1-k_2)T_1^+(k_1)T_2^+(k_2) = T_2^+(k_2)T_1^+(k_1)S_{12}(k_1-k_2)\,, \tag{2.6}$$

$$R^+(-k)R^+(k)+T^-(k)T^+(k) = \mathbb{I} \quad , \quad R^-(k)T^+(k)+T^+(-k)R^+(k) = 0\,. \tag{2.7}$$

We refer the reader to the original works for the remaining relations. What is important here is to note that the relations involving S_{12} and T^+ are *cubic*, in that they involve three terms on each side.

Now, let us move on to the equations obtained from the second approach. The physical data is given by $\mathcal{S}_{AB}(k_1,k_2)$, $\mathcal{R}(k)$, $\mathcal{T}(k)$ (where we labelled the auxiliary spaces by letters for later convenience). One may wonder about the possibility of different left and right reflection and transmission but we will see in the example below that this is intrinsically encoded in this approach. The sought relations appear as Fock representations of the RT algebra [17] and read

$$\begin{aligned} \mathcal{S}_{AB}(k_1,k_2)\mathcal{R}_A(k_1)\mathcal{S}_{BA}(k_2,-k_1)\mathcal{R}_B(k_2) \\ = \mathcal{R}_B(k_2)\mathcal{S}_{AB}(k_1,-k_2)\mathcal{R}_B(k_1)\mathcal{S}_{BA}(-k_2,-k_1)\,, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mathcal{S}_{AB}(k_1,k_2)\mathcal{R}_A(k_1)\mathcal{S}_{BA}(k_2,-k_1)\mathcal{T}_B(k_2) \\ = \mathcal{T}_B(k_2)\mathcal{S}_{AB}(k_1,k_2)\mathcal{R}_A(k_1)\mathcal{S}_{BA}(k_2,-k_1)\,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \mathcal{S}_{AB}(k_1,k_2)\mathcal{T}_A(k_1)\mathcal{S}_{BA}(k_2,k_1)\mathcal{T}_B(k_2) \\ = \mathcal{T}_B(k_2)\mathcal{S}_{AB}(k_1,k_2)\mathcal{T}_A(k_1)\mathcal{S}_{BA}(k_2,k_1)\,, \end{aligned} \tag{2.10}$$

$$\mathcal{R}(k)\mathcal{R}(-k)+\mathcal{T}(k)\mathcal{T}(k) = \mathbb{I} \quad , \quad \mathcal{R}(k)\mathcal{T}(-k)+\mathcal{T}(k)\mathcal{R}(k) = 0\,. \tag{2.11}$$

Here the equations involving transmission are *quartic*. It seems that the two approaches are incompatible as long as one is interested in the physical equations for reflection and transmission. But how to decide which one is correct?

One can notice that both approaches agree on the reflection equation ((2.4) and (2.8)) and on the unitarity relations ((2.7) and (2.11)). Forgetting the algebraic setup, one can think of starting from these equations to get some insight. This has been first presented in [17] and detailed in [21]. It is argued that solving (2.11) for $\mathcal{T}(k)$ in terms of $\mathcal{R}(k)$ and using the reflection equation, one gets that $\mathcal{T}(k)$ must satisfy the quartic relations (2.9) and (2.10) and not the cubic ones.

2.3 Reconciling points of view

The above conclusions were at the heart of an apparent controversy. In [21], the situation has been fully clarified. It appears that there is *no contradiction* since the first approach is merely a particular case of the second one. Indeed, making the following particular choice for the scattering data of the second approach*

$$\mathcal{S}_{AB}(k_1, k_2) = \left(\begin{array}{cccc} S_{12}(k_1 - k_2) & 0 & 0 & 0 \\ 0 & \mathbb{I} \otimes \mathbb{I} & 0 & 0 \\ 0 & 0 & \mathbb{I} \otimes \mathbb{I} & 0 \\ 0 & 0 & 0 & S_{21}(k_2 - k_1) \end{array} \right) . \tag{2.12}$$

$$\mathcal{R}(k) = \left(\begin{array}{cc} R^+(-k) & 0 \\ 0 & R^-(-k) \end{array} \right) \quad , \quad \mathcal{T}(k) = \left(\begin{array}{cc} 0 & T^-(k) \\ T^+(k) & 0 \end{array} \right) , \tag{2.13}$$

and plugging in (2.8-2.11), one recovers the equations for the data of the first approach.

Conclusions

The reflection-transmission quantum Yang-Baxter equations have the form (2.8-2.11). They are naturally obtained in the Reflection-Transmission (RT) algebras approach [17] and reproduce as a particular case the original equations obtained in [16, 18]. However, they do not suffer from the same limitation and allow for quantum *interacting* integrable systems with reflection *and* transmission. This was first illustrated in [19] using the well-known nonlinear Schrödinger equation. It appears that the way out of free models is *not* due to the more general dependence in the momenta allowed for in the RT approach. This is discussed and illustrated in [21] where explicit solutions with a Lorentz invariant two-body scattering matrix are established. The key point lies in the quartic relations involving transmission. In this respect, we remark that the physical data used to solve the nonlinear Schrödinger equation with impurity satisfies the quartic but not the cubic relations.

This shows that the RT algebra approach offers new interesting possibilities in the study of integrable systems with impurity.

*Here, if the labels 1 and 2 correspond to the auxiliary space $\mathbb{C}^n$ then the labels A and B correspond to the auxiliary space $\mathbb{C}^2 \otimes \mathbb{C}^n$.

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